Introduction to Graph Theory

Introduction
These notes are primarily a digression to provide general background remarks. The subject is an efficient procedure for the determination of voltages and currents of a given network. A network comprised of B branches involves 2B unknowns, i.e., each of the branch voltages and currents. However the branch volt-ampere relations of the network, presumed to be known, relate the current and the voltage of each branch. Hence a calculation of either B currents or B voltages (or some combination of B voltages and currents), and then substitution in the B branch volt-ampere relations, provides all the voltages and currents.

In general however neither the B branch voltages nor the B branch currents are independent, i.e., some of the B voltage variables for example can be expressed as a combination of other voltages using KVL, and some of the branch currents can be related using KCL. Hence there generally are fewer than B independent unknowns. In the following notes we determine the minimum number of independent variables for a network analysis, the relationship between the independent and dependent variables, and efficient methods of obtaining independent equations to determine the variables. In doing so we make use of the mathematics of Graph Theory.

Graph Theory
A circuit graph is a description of the just the topology of the circuit, with details of the circuit elements suppressed. The graph contains branches and nodes. A branch is a curve drawn between two nodes to indicate an electrical connection between the nodes.

A directed graph is one for which a polarity marking is assigned to all branches (usually an arrow) to distinguish between movement from node A to B and the converse movement from B to A.

A connected graph is one in which there is a continuous path through all the branches (any of which may be traversed more than once) which touches all the nodes. A graph that is not connected in effect has completely separate parts, and for our purposes is more conveniently considered to be two (or more) independent graphs.

We consider a directed, connected graph. An electrical circuit and its graph are illustrated to the right. The graph retains only topological information about the circuit.

For reasons to appear branches are divided into two topological groups, tree branches and links. A tree of the graph is a subset of the branches such that all graph nodes are connected by branches but without forming a closed path. These branches then are the tree branches. The remaining branches (collectively called a co-tree) are the links. A tree and its links for the illustrative circuit are shown in the figure (labeled T and L respectively).

For a network with B branches and N nodes the number of tree branches is N–1; this is the minimum number of branches needed to connect N nodes, and connecting another branch will form a loop. In general a graph contains many different trees, but for each of these trees there are N–1 tree branches. It follows then that there are B–N+1 links.
Choosing Independent Current Variables:
Given a network graph with B branches and N nodes select a tree, any one will do for the present purpose. Remove all the link branches so that, by definition, there are no loops formed by the remaining tree branches. It follows from the absence of any closed paths that all the branch currents become zero. Hence by 'controlling' just the link currents all the branch currents can be controlled. This control would not exist in general using fewer than all the link branches because a loop would be left over; depending on the nature of the circuit elements branches making up the loop current could circulate around the loop. Using more than the link branches is not necessary. Hence it should be possible to express all the branch currents in terms of just the link currents, i.e., there are B-N+1 independent current variables, and link currents provide one such set of independent variables.

A straightforward formal procedure for describing tree branch currents in terms of link currents is illustrated as follows; the graph on the left side of the figure below is used for illustration.

The first step is selection of a tree, e.g., the tree on the right of the figure above. The tree makes clear that a set of independent current variables is formed using link branches 2, 4, 5, and 7. (Branch numbering and specification of the branch polarity arrows is more or less arbitrarily.)

Select a link; start with any one since each link is used in turn. Remove any other link used previously and then insert the single selected link to the tree to form a single loop. For example inserting link branch 2 defines a loop formed by branches 2, 3, 6, and 8. Write a KVL equation for this loop.

As an aid for keeping track of information, particularly for larger graphs, we can use a matrix such as that shown below. The rows of the table define the N–B+1 loops, four for the illustration, which will be formed by the links. Rows are labeled by the ID for the link branch used to form the loop, since this branch is unique for the loop so designated. The columns of the matrix correspond to the full set of graph branches. Since it is generally helpful to be systematic in these matters we choose to write all the KVL loop equations circulating about a loop in the direction of the link arrow. The loop equation will consist of a sum of loop branch voltage drops; the coefficient of a branch voltage drop will be +1 if circulation around the loop is in the same direction as the branch arrow, and –1 otherwise. Enter the coefficients into the table, row by row. Branches that are not in the loop for a given row will have no entry (or enter 0), indicating their lack of involvement in a particular loop.

<table>
<thead>
<tr>
<th>Loop</th>
<th>Branch</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 -1</td>
</tr>
<tr>
<td>4</td>
<td>-1 1</td>
</tr>
<tr>
<td>5</td>
<td>1 1</td>
</tr>
<tr>
<td>7</td>
<td>1 1</td>
</tr>
</tbody>
</table>
For example the loop formed by inserting link branch 2 involves branches 2, 3, 6, and 8. Circulation about the loop CCW (as suggested by the link branch polarity arrow) passes though branch 6 in the same direction as the arrow for that branch, and passes in the reverse direction to the arrow for the other two branches.

The branches forming a row of the matrix, i.e., the branches forming a loop for the link associated with the row, are called a 'tieset' for historical reasons. The independence of the KVL equations is assured because each loop involves at least one branch, a link, not associated with any other of the loops. Therefore no one loop equation can be obtained by a combination of the other loop equations.

This 'tieset' procedure guarantees that the maximum number of independent KVL loop equations will be written directly and efficiently. No guessing or trial and error is involved. But in fact this procedure provides even more information; it provides in a similar assured and efficient manner the maximum number of independent KCL equations. This comes about in the following manner.

a) The matrix rows are identified by the link inserted to form a loop with tree branches; the tree branches completing the loop are identified by the column entries for the given row.
b) It follows that the row entries for a given column indicate the links that involve the use of the column branch to form a loop.
c) From b) it follows that any closed path (loop) from one end of the column branch to the other (see diagram to right) must pass through the links which necessarily are part of the loops (one link per loop). Further the polarity of each link relative to the column branch is indicated by the sign of the column entry. Hence a KCL equation can be written using the column branch and the column links; these branches correspond to all the currents leaving (or entering) a closed volume.
d) The equations so written are independent, since each contains a branch (the column branch) not contained in any other equation. Hence the link currents form the minimum number of independent current variables.

For the circuit illustrated, \( I_1 = -I_4 + I_5 \), \( I_3 = -I_2 + I_7 \), \( I_6 = I_2 - I_4 + I_5 \), and \( I_8 = -I_2 + I_4 + I_7 \).
e) The general conclusions can be verified in a specific instance very easily. Simply select a tree (column) branch and then reduce the length of all other tree branches to zero thus bringing their nodes together. This separates all the nodes into two groups clustered about the ends of the selected tree branch. To cross over from one of these node groups to the other requires using either the selected tree branch or a link. The figure to the left shows the nodes 'collapsed' on tree branch #6, and the branches that cross from one set of nodes to the other. Note that not all the links cross over.
In summary then:
  a) Select a tree and form the tieset matrix; the matrix columns correspond to all the circuit branches while the rows correspond to the link branches. In general it is easiest to fill in the matrix row by row; each row corresponds to a KVL equation written around a loop formed when the link corresponding to the row is added to the tree (all other links absent). There are B–(N-1) KVL equations expressed in terms of tree branch voltages; these are independent equations since each equation involves a link voltage not involved in any other equation.
  b) From the matrix columns write N-1 KCL equations expressing the tree branch currents in terms of the link currents. The link currents thus form a set of variables by which all branch currents can be expressed.
  c) In addition to these equations there are B constitutive equations relating a branch voltage to a branch current.

To solve these equations:
  a) Substitute for the branch voltages in the KVL equations to obtain B-(N-1) equations in terms of B branch current variables;
  b) Substitute from the N-1 KCL equations to eliminate all but the B-(N-1) link currents;
  c) Solve the B -(N-1) equations for the link current unknowns;
  d) Back substitute to calculate the remaining branch voltages and currents.

While this may be a tedious procedure (typically it is) it has a number of rather important advantages. We are assured from the beginning that the procedure will converge to a solution, and that there is no uncertainty about the independence of the equations selected. Moreover for topologically complicated circuits filling in the matrix rows is simplified because only one loop is formed for each row.

In summary then:
  a) Select a tree and form the tieset matrix; the matrix columns correspond to all the circuit branches while the rows correspond to the link branches. In general it is easiest to fill in the matrix row by row; each row corresponds to a KVL equation written around a loop formed when the link corresponding to the row is added to the tree (all other links absent). There are B–(N-1) KVL equations expressed in terms of tree branch voltages; these are independent equations since each equation involves a link voltage not involved in any other equation.
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As an additional illustration the KVL and KCL equations are written below for same circuit graph used before, but with a different tree.

\[ v_2 + v_6 + v_7 = 0 \]
\[ v_1 + v_3 - v_4 + v_6 + v_7 = 0 \]
\[ v_1 + v_5 + v_6 = 0 \]
\[ -v_1 + v_4 - v_6 + v_8 = 0 \]
\[ i_1 = i_3 + i_5 - i_8 \]
\[ i_2 = i_2 \]
\[ i_3 = i_3 \]
\[ i_4 = -i_3 + i_8 \]
\[ i_5 = i_5 \]
\[ i_6 = i_2 + i_3 + i_5 - i_8 \]
\[ i_7 = i_2 + i_3 \]
Choosing Independent Voltage Variables:

Voltages and current appear in the KCL and KVL in a formally symmetric fashion, and so it should not be surprising to find a procedure employing voltage variables dual to the tieset procedure for current variables. Thus consider again a network with B branches and N nodes, and select a tree for the graph. For an illustration we use the same graph as before and, although it is not necessary, for comparison to the preceding discussion we use the same tree as before.

Whereas the link currents provide a set of independent current variable the N-1 tree branch voltages provide a set of independent voltage variables; setting these branch voltages to zero forces all branch voltages to zero. Remember that each link is a connection between nodes, and hence KVL enables each link voltage to be expressed in terms of only tree branch voltages. It is convenient to record these expressions in a (‘cutset’) matrix similarly to what was done for the current variables with a ‘tieset’ matrix. The rows of the matrix are associated with the voltage drop across tree branches (in place of link currents for the tieset matrix) and the columns are the graph branches. A column entry in a given row is ' ', 1, or -1, determined as follows. Insert each branch into the tree in turn; remove branches inserted earlier. Express the voltage drop across the inserted branch in terms of tree branch voltages. This is always possible since the inserted branch either is a tree branch, or it is a link and forms a loop with tree branches. The table for the illustrative network and the tree selected is shown to the left; the coefficients of the voltage expressions are entered into the table. The expressions for branch voltage drop in terms of tree branch voltages can be read back from the columns, e.g.,

\[ v_4 = v_1 + v_6 - v_8 \]

The rows of the matrix provide very useful information also; the group of branches corresponding to nonzero entries in a row is called a cutset. The voltage expression for each of the cutset branches requires use of the tree branch voltage corresponding to the row. Suppose all the tree branches shrink to zero length except the one corresponding to the row. This divide the graph nodes into two group such that a path from a node in one group to a node in the other group must pass through a cutset branch (see figure to right). Hence the cutset branches can be used to write KCL equations.

For example for the cutset associated with tree branch 8 write

\[ v_4 = v_1 + v_6 - v_8 \]
\[ 0 = i_2 - i_4 - i_7 + i_8 \]

These KCL equations must be independent since each equation involves a tree branch not included in any other equations.

In summary then:

a) Select a tree and form the cutset matrix; the matrix columns correspond to all the circuit branches while the rows correspond to the tree branches. In general it is easiest to fill in the matrix column by column; each column corresponds to a KVL equation written to express each branch voltage in terms of tree branch voltages. The tree voltages form a set of voltage variables by which all branch voltages can be expressed. There are \( B-(N-1) \) KVL equations for the link branch voltages expressed in terms of tree branch voltages; these are independent equations since each equation involves a link voltage not involved in any other equation.

b) From the matrix rows write \( N-1 \) independent KCL equations; the equations are independent because each equation involves a unique tree voltage. In addition to these equations there are \( B \) constitutive equations relating a branch voltage to a branch current.

To solve these equations:

a) Substitute for the branch currents in the KCL equations to obtain \( N-1 \) equations in terms of \( B \) branch voltage variables;

b) Substitute from the \( B-(N-1) \) KVL equations to eliminate all but the \( N-1 \) tree branch voltages.

c) Solve the \( N-1 \) equations for the link current unknowns;

d) Back substitute to calculate the remaining branch voltages and currents.
A circuit graph (directed, connected) and one of its trees is drawn to the left. A tree analysis (link currents as current variables) follows. The graph has 10 branches and 5 nodes. There are therefore 4 tree branches and 6 links. The number of independent current variables, equal to the number of links, is 6.

In general, the tie set matrix is filled in easiest by rows. Add a link to the tree, one link at a time, and write a KVL equation for the loop so formed. For simplicity and clarity the matrix to the right is filled in always circulating in the direction of the link direction arrow. Only the coefficients of the branch voltages are entered; +1 if the circulation is in the direction of the branch arrow, -1 if it is opposite. Note that although the circuit graph has a modest topological complexity the loop formed by each link is very evident.

\[
\begin{align*}
\nu_3 + \nu_7 - \nu_8 &= 0 \\
\nu_1 + \nu_2 + \nu_4 - \nu_7 + \nu_8 &= 0 \\
\nu_1 + \nu_2 + \nu_5 - \nu_7 &= 0 \\
\nu_2 + \nu_6 - \nu_8 &= 0 \\
-\nu_1 - \nu_2 + \nu_9 &= 0 \\
\nu_2 - \nu_7 + \nu_8 + \nu_{10} &= 0
\end{align*}
\]

The KVL equations are shown to the left, these can easily be read back from the matrix. Note that the equations are independent of one another since each involves a voltage, a link voltage, not involved in any other equation.

A tree branch divides the graph nodes into two groups, and the matrix column corresponding to that tree branch reveals the links which connect one group to the other. On applying KCL the tree branch current can be expressed in terms of those link currents. The appropriate coefficients are indicated by the matrix entries. The current expressions for the illustration are shown to the right. Note that the equations are independent of one another; each equation involves a branch current not included in any other equation.

In general the branch volt-ampere relations would be known. These can be used to substitute branch currents for branch voltages in the KVL equations. The KCL equations then can be substituted to eliminate the tree branch currents, leaving 6 equations involving the 6 link currents. Solve this set of linear simultaneous equations for the link currents and back-substitute to determine the remaining branch currents and voltages.
CUTSET EXAMPLE

This is the same graph and tree as before; however now a cutset analysis (tree voltages as voltage variables) follows. As before the graph has 10 branches and 5 nodes. There are therefore 4 tree branches and 6 links. The number of independent voltage variables, equal to the number of tree branches, is 4.

\[
\begin{array}{c|cccccccccc}
\text{cutset matrix} \\
\hline
\text{tree branch} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\
2 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\
7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
\end{array}
\]

The tree branch voltages provide a set of voltage variables in terms of which all the branch voltages can be expressed. Note that the KVL equations are independent of one another since each involves a branch voltage, the link voltage, which is not involved in any other equation. The voltage equations, shown to the left, can be read back directly from the matrix.

Each tree branch separates the graph nodes into two groups which are connected through links. The sketch to the right indicates the separation for tree branch #3; the nodes are grouped by imagining the tree branches to shrink to zero length. The links connecting the node groups for a given tree branch (and the tree branch also) are indicated by each row entry; the matrix entries also indicate the appropriate coefficients. The KCL equations corresponding to the rows are shown below:

\[
\begin{align*}
i_1 - i_4 - i_5 + i_9 &= 0 \\
i_2 - i_4 - i_5 - i_6 + i_9 - i_{10} &= 0 \\
i_3 + i_4 + i_5 + i_6 + i_7 + i_{10} &= 0 \\
i_3 - i_4 + i_8 - i_{10} &= 0
\end{align*}
\]
ILLUSTRATIVE EXAMPLE

Example Circuit

Circuit Graph

Graph Tree

CutSet matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
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<td></td>
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<tr>
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<td></td>
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<td></td>
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<td></td>
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<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
    v_2 &= -v_1 + v_6 + v_7 \\
    v_4 &= -v_3 - v_5 - v_6 - v_7 \\
    v_8 &= v_3 + v_7
\end{align*} \]

\[ \begin{align*}
    i_1 - i_2 &= 0 \\
    i_3 - i_4 + i_8 &= 0 \\
    i_5 - i_4 &= 0 \\
    i_6 + i_2 - i_4 + i_8 &= 0
\end{align*} \]

Branch volt-ampere relations; the branch arrow is used for both the voltage and current polarity markers.

\[ \begin{align*}
    v_1 &= -1 \\
    v_5 &= -3 \\
    v_2 &= 12 \\
    v_6 &= 216 \\
    v_3 &= 313 \\
    v_7 &= 217 \\
    v_4 &= 14 \\
    v_8 &= 218
\end{align*} \]

Substitute from equations 3 into equations 2, replacing branch currents by branch voltages.

Note that this cannot be done (yet) for the voltage source branches.

Use equations 1 to replace the link voltages in equations 4 by the independent tree branch voltage variables.

\[ \begin{align*}
    v_3 - 3v_4 + 1.5v_8 &= 0 \\
    i_5 - v_4 &= 0 \\
    v_6 + 2v_2 - 2v_4 &= 0 \\
    v_7 + 2v_2 - 2v_4 + v_8 &= 0
\end{align*} \]

Note that two of the equations involve the unknown source branch currents. However, this is no problem. Simply discard these equations (use them later to calculate the currents) and replace them with the two equations \( v_1 = -1 \) and \( v_5 = -3; v_1 \) and \( v_5 \) are really not unknowns since the source strength specifies their values.

Solve for the independent current variables (use Cramer's rule or Gaussian elimination) and back-substitute to calculate the other branch voltages and currents. Check calculations by verifying KVL, KCL for a few samples.

\[ \begin{align*}
    v_1 &= -1 V \\
    i_1 &= 0.93 A \\
    v_2 &= 0.93 V \\
    i_2 &= 0.93 A \\
    v_3 &= 1.85 V \\
    i_3 &= 0.62 A \\
    v_4 &= 1.22 V \\
    i_4 &= 1.22 A \\
    v_5 &= -3 V \\
    i_5 &= 1.22 A \\
    v_6 &= 0.57 V \\
    i_6 &= 0.29 A \\
    v_7 &= -0.64 V \\
    i_7 &= -0.32 A \\
    v_8 &= 1.21 V \\
    i_8 &= 0.60 A
\end{align*} \]
Circuits Graph Tree

Example Circuit

TieSet matrix

<table>
<thead>
<tr>
<th>link</th>
<th>branch</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1  1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1  1  1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

KCL equations for loops formed by adding links to the tree:
\[ v_1 + v_2 - v_6 - v_7 = 0 \]
\[ v_3 + v_4 + v_5 + v_6 + v_7 = 0 \]
\[ -v_3 - v_7 + v_8 = 0 \]

KCL expressions (other than identities) written from the matrix columns. This expresses the tree branch currents in terms of the independent link current variables; note that this cannot be done for the source branch currents yet.

Branch volt-ampere relations; the branch arrow is used for both voltage and current polarity markers. Note that the source branch relations do not involve current.

\[ v_1 = -1 \]
\[ v_2 = 12 \]
\[ v_3 = 31 \]
\[ v_4 = 14 \]
\[ v_5 = -3 \]
\[ v_6 = 216 \]
\[ v_7 = 217 \]
\[ v_8 = 218 \]

Substitute equations 3 into equations 1, replacing voltages by currents (or, for source branches, by the voltage source strength).

\[-1 + i_2 - 2i_6 - 2i_7 = 0 \]
\[3i_3 + i_4 - 3 + 2i_6 + 2i_7 = 0 \]
\[-3i_3 - 2i_6 + 2i_8 = 0 \]

Use the equations 2 to replace the tree branch currents in equations 4 by the independent link current variables, so obtaining three independent equations in three unknowns.

\[ 1 = 5i_2 - 4i_4 + 2i_8 \]
\[ 3 = -4i_2 + 8i_4 - 5i_8 \]
\[ 0 = 2i_2 - 5i_4 + 7i_8 \]

\[ \begin{bmatrix} 1 & 5 & -4 & 2 \\ 3 & -4 & 8 & -5 \\ 0 & 2 & -5 & 7 \end{bmatrix} \begin{bmatrix} i_2 \\ i_4 \\ i_8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \]

Solve for the independent current variables (use Cramer's rule or Gaussian elimination) and back-substitute to calculate the other branch voltages and currents. Check calculations by verifying KVL, KCL for a few samples.

\[ v_1 = -1 Y \]
\[ v_2 = 0.93 Y \]
\[ v_3 = 1.85 Y \]
\[ v_4 = 1.22 Y \]
\[ v_5 = -3 Y \]
\[ v_6 = 0.57 Y \]
\[ v_7 = -0.64 Y \]
\[ v_8 = 1.21 Y \]

\[ i_2 = 0.93 A \]
\[ i_3 = 0.62 A \]
\[ i_4 = 1.22 A \]
\[ i_5 = 1.22 A \]
\[ i_6 = 0.29 A \]
\[ i_7 = -0.32 A \]
\[ i_8 = 0.60 A \]