Example 3: During some consecutive days the team played 14 games?

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games.
  - $a_j$: the number of games played on or before day $j$ ($j \in [30]$)
  - $a_1, \ldots, a_{30}, a_1 + 14, \ldots, a_{30} + 14$ are in $[1, 59]$
  - There exist $i, j \in [30]$ such that $a_i = a_j + 14$
  - The number of games on day $j + 1, \ldots, i$ is 14, exactly
Example

Definition: Let $a_1, \ldots, a_N$ be a sequence of numbers

- A subsequence is a sequence of the form $a_{i_1}, \ldots, a_{i_m}$
- strictly increasing: $a_{i_{j+1}} > a_{i_j}$ for $j = 1, 2, \ldots, m - 1$
- strictly decreasing: $a_{i_{j+1}} < a_{i_j}$ for $j = 1, 2, \ldots, m - 1$

Example 4: Let $a_1, \ldots, a_{n^2+1}$ be a sequence of distinct real numbers.

- There is a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.
  - $8, 11, 9, 1, 4, 6, 12, 10, 5, 7$ contains $10 = 3^2 + 1$. 1 4 6 7
Example

• $i_j$: the length of longest increasing subsequence $a_j$, ...
• $d_j$: the length of longest decreasing subsequence $a_j$, ...
• If $i_j, d_j \leq n$ for every $j$, then $\exists s, t$ such that $i_s = i_t, d_s = d_t$
  • $a_s < a_t \Rightarrow i_s \geq i_t + 1$ contradiction
  • $a_s > a_t \Rightarrow d_s \geq d_t + 1$ contradiction
Example

Example 5: There is a subsequence whose summation is divisible by $m$?

- Let $a_1, \ldots, a_m$ be a sequence of positive integers.
- Then there are integers $k, l$ such that
  - $1 \leq k \leq l \leq m$ and
  - $m | (a_k + a_{k+1} + \cdots + a_l)$
    - Define $T_i = a_1 + \cdots + a_i$ for every $i \in [m]$
    - If $\exists i_0 \in [m]$ s.t. $m | T_{i_0}$, the proof is done ($k = 1, l = i_0$)
    - Otherwise, $\{T_1, \ldots, T_m\} \mod m \subseteq \{1,2,\ldots,m-1\}$
      - $\exists i, j \in [m]$ s.t. $T_i \mod m = T_j \mod m$
      - $a_{i+1} + \cdots + a_j = T_j - T_i$ is a multiple of $m$
      - $k = i + 1, l = j$: the proof is done
Example

Example 6: There is a subsequence with given summation?

- Let \( a_1, \ldots, a_{100} \) be a sequence of 1’s and 2’s.
- Suppose that \( a_i + a_{i+1} + \cdots + a_{i+9} \leq 16 \) for every \( i \in [91] \)
- Then there are integers \( k, l \) such that
  - \( 1 \leq k \leq l \leq 100 \) and \( a_k + a_{k+1} + \cdots + a_l = 39 \)
    - Define \( T_i = a_1 + \cdots + a_i \) for every \( i \in [100] \)
    - Then \( T_i \leq 160 \) for every \( i \in [100] \)
    - Consider \( T_1, \ldots, T_{100}, T_1 + 39, \ldots, T_{100} + 39 \)
      - Each element \( \leq 10 + 39 = 199 \)
    - There exists \( i, j \in [100] \) such that \( T_i = T_j + 39 \)
    - \( a_{j+1} + \cdots + a_i = 39 \)
    - Define \( k = j + 1 \) and \( l = i \)
Example 7:

- Divide the set $S = \{1, 2, \ldots, 67\}$ into 4 pairwise disjoint subsets $S_1, S_2, S_3, S_4$ such that $S = S_1 \cup S_2 \cup S_3 \cup S_4$.
- Then one of the subsets must contain $x, y, z$ s.t. $z = x - y$
Proof by contradiction: none of the subsets has that property.

Suppose that $|S_1| \geq 17$. Say $S_1 \supseteq \{a_1, ..., a_{17}\}$.

- $b_1 := a_2 - a_1, b_2 := a_3 - a_1, ..., b_{16} := a_{17} - a_1 \notin S_1$
- $B := \{b_1, ..., b_{16}\} \subseteq S$

$|B \cap S_2| \geq 6$ or $|B \cap S_3| \geq 6$ or $|B \cap S_4| \geq 6$

- say $|B \cap S_2| \geq 6$ and $B \cap S_2 \supseteq \{b_1, ..., b_6\}$
- $c_1 := b_2 - b_1, c_2 := b_3 - b_1, ..., c_5 := b_6 - b_1 \notin S_1 \cup S_2$
- $C := \{c_1, ..., c_5\}$

$|C \cap S_3| \geq 3$ or $|C \cap S_4| \geq 3$

- say $|C \cap S_3| \geq 3$ and $C \cap S_3 \supseteq \{c_1, c_2, c_3\}$
- $d_1 := c_2 - c_1, d_2 := c_3 - c_1 \notin S_1 \cup S_2 \cup S_3$
- $d_1, d_2 \in S_4$
- $d_2 - d_1 \notin S_1 \cup S_2 \cup S_3 \cup S_4$
Example

Example 8: Ramsey Number:

- There is a group of 6 people, each pair of them are either friends or enemies. Show that there are either 3 mutual friends or three mutual enemies in the group.
  - Suppose that the 6 persons are A,B,C,D,E,F
  - A has either \( \geq 3 \) friends or enemies (pigeonhole principle)
  - Suppose A has B, C, D as friends
    - If any 2 of B, C, and D are friends, then they three are friends, done
    - If B, C and D are enemies, done
Theorem: Let $A, B$ be any sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.

- $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
- $A \setminus B, A \cap B, B \setminus A$ are pairwise disjoint
- $|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|$

\[= |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B|\]
Principle of IE (Two Sets)

Example:

• How many positive integers ≤ 1000 are divisible by 7 or 11?
  • $A = \{n \leq 1000: 7|n\}, B = \{n \leq 1000: 11|n\}$
  • $|A \cup B| = |A| + |B| - |A \cap B|$
    
    $= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{77} \right\rfloor = 220$
Principle of IE (Two Sets)

Example:

- Decide the number of the permutations of $a, b, c, d, e, f$ that do not contain $ace$ or $df$
  - $A := \{\text{all permutations of } a, b, c, d, e, f\}$
  - $A_1 := \{\text{permutations containing } ace\}$
  - $A_2 := \{\text{permutations containing } df\}$
  - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 4! + 5! - 3! = 138$
  - $|A - A_1 \cup A_2| = 6! - 138 = 582$
Principle of IE (Three Sets)

**Theorem:** Let $A, B, C$ be any sets. Then

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|

- $|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|
  = |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$

- $|(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$
Principle of IE (Three Sets)

Example: 2092 students are learning Spanish, French, and Russian

- 1232 students are learning Spanish
- 879 students are learning French
- 114 students are learning Russian
- 103 are learning both Spanish and French
- 23 are learning both Spanish and Russian
- 14 are learning both French and Russian
- # of students learning all three languages
  - $A = \{\text{Spanish}\}; \quad B = \{\text{French}\}, \quad C = \{\text{Russian}\};$
  - $|A \cup B \cup C| = 2092$
  - $|A| = 1232, \quad |B| = 879, \quad |C| = 114;$
  - $|A \cap B| = 103, \quad |A \cap C| = 23, \quad |B \cap C| = 23;$
  - $|A \cap B \cap C| = 7$
Principle of IE (Three Sets)

Example:

- Decide $|S|$ for $S = \{s \in \{a, b, c, d\}^n : \text{each of } a, b, c \text{ appears } \geq 1 \text{ time in } s\}$
  - $S_a := \{s \in \{a, b, c, d\}^n : a \text{ does not appear in } s\}$
  - $S_b := \{s \in \{a, b, c, d\}^n : b \text{ does not appear in } s\}$
  - $S_c := \{s \in \{a, b, c, d\}^n : c \text{ does not appear in } s\}$
  - $S = \{a, b, c, d\}^n \setminus (S_a \cup S_b \cup S_c)$
  - $|S_a \cup S_b \cup S_c| = 3^n + 3^n + 3^n - 2^n - 2^n - 2^n + 1$
  - $|S| = 4^n - 3^{n+1} + 3 \cdot 2^n - 1$