Discrete Mathematics
Graph Theory

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Planar Graphs

Definition: Let \( G = (V, E) \) be a graph.

- **planar**: \( G \) can be drawn in the plane without any edges crossing.

Applications

- design of electronic circuits and road networks
Example: \( K_{3,3} \) is not planar.
Example

Example: \( K_5 \) is not planar.
Example
**Euler’s Formula**

**Theorem:** Let $G$ be a *connected planar simple* graph

- $e$: # of edges in $G$
- $v$: # of vertices in $G$
- $r$: # of regions in a planar representation of $G$.
  - Then $r = e - v + 2$.

6 regions from the planar representation of a graph
Euler’s Formula

• Construct a sequence of graphs $G_1, ..., G_e$
  • $G_1$ has one edge
  • for $i = 2$ to $n$ do
    • find edge $\in E$, edge $= \{u, v\} \not\in E(G_{i-1}), \{u, v\} \cap V(G_{i-1}) \neq \emptyset$
    • $G_i = G_{i-1} + \text{edge}$
  • $v_i := |V(G_i)|, e_i := |E(G_i)|, r_i = |R(G_i)|$ for $i \in [e]$
• Show that $r_i = e_i - v_i + 2$ for every $i$
  • $i = 1$: $r_i = 1, e_i = 1, v_i = 2$
  • $i = 2$: $r_i = 1, e_i = 2, v_i = 3$
• Suppose that $r_i = e_i - v_i + 2$.
  • $r_{i+1} = e_{i+1} - v_{i+1} + 2$
    • Let $e_{i+1} = \{u, v\}: \{u, v\} \subseteq V(G_i), \ |\{u, v\} \cap V(G_i)| = 1$
    • $\{u, v\} \subseteq V(G_i): r_{i+1} = r_i + 1; e_{i+1} = e_i + 1; v_{i+1} = v_i$
    • $|\{u, v\} \cap V(G_i)| = 1: r_{i+1} = r_i, e_{i+1} = e_i + 1, v_{i+1} = v_i + 1$
  • In both cases, $r_{i+1} = e_{i+1} - v_{i+1} + 2$
Euler’s Formula

**Corollaries:** Let $G$ be a connected planar simple graph.

- If $v = |V(G)| \geq 3$, then $e := |E(G)| \leq 3v - 6$.
  - Let $P_1, \ldots, P_r$ be the regions obtained from $G$
  - $N_i := \#$ of edges in $E$ that are on the border of $P_i$
    - $N_i \geq 3$ for every $i \in [r]$
  - $M_j := \#$ of times of the $j$th edge being counted
    - $M_j = 2$
  - $3r \leq N_1 + \cdots + N_r = M_1 + \cdots + M_e = 2e$
  - $r = e - v + 2$
Corollaries: Let $G$ be a connected planar simple graph.

- There is a vertex $u$ such that $\text{deg}(u) \leq 5$.
  - When $G$ has $< 3$ vertices, the statement is true.
  - When $G$ has $\geq 3$ vertices, $\text{deg}(u) \geq 6, \forall u \in V \Rightarrow 2e = \sum u \text{deg}(u) \geq 6v$
    - However, $e \leq 3v - 6$ must hold. A contradiction appears.
- $v = |V(G)| \geq 3$ and $\not\exists$ circuits of length 3, then $e \leq 2v - 4$.
  - $N_i \geq 4$ for every $i \in [r]$
  - $4r \leq N_1 + \cdots + N_r = M_1 + \cdots + M_e = 2e$
  - $r = e - v + 2$
  - $e \leq 2v - 4$
How to Decide Planar Graphs

Definition: Let $G = (V, E)$ be a graph and $\{u, v\} \in E$.

- **elementary subdivision**: $G' = (V \cup \{w\}, E - \{u, v\} + \{u, w\} + \{v, w\})$
- **homeomorphic**: two graphs can be obtained from the same graph via elementary subdivisions
Kuratowski's theorem

Theorem: Let $G = (V, E)$ be a graph.
- $G$ is nonplanar iff it has a subgraph homeomorphic to $K_{3,3}$ or $K_5$. 
Kuratowski's theorem

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Graph Coloring

**Definition:** Let $G = (V, E)$ be a simple graph.

- **$k$-coloring:** a map $f : V \to [k]$ s.t. $\{u, v\} \in E \Rightarrow f(u) \neq f(v)$
- **chromatic number ($\chi(G)$):** the least $k$ s.t. $G$ has a $k$-coloring.

\[
\chi(G) = 3
\]

If $G$ has subgraph isomorphic to $K_t$, then $\chi(G) \geq t$
**Graph Coloring**

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\[
\chi(G) = 4
\]
Graph Coloring

Example:

- $\chi(K_{m,n}) = 2$, 
- $\chi(K_n) = n$
- $\chi(C_n) = 2$ if $2|n$; $\chi(C_n) = 3$ if $2|(n - 1)$
The Five Color Theorem

The Four Color Theorem: If $G$ is planar, then $\chi(G) \leq 4$.

The Five Color Theorem: If $G$ is planar, then $\chi(G) \leq 5$.

- $|V| \leq 5 \Rightarrow \chi(G) \leq 5$
- Suppose that $\chi(G) \leq 5$ when $|V| \leq k$
- Need to prove for $|V| = n = k + 1$
- $\exists \ v \in V, \deg(v) \leq 5$
- $H := G - v$ has a 5-coloring $c : V(H) \rightarrow [5]$
- If $|\{c(u) : u \in N(v)\}| \leq 4$, $c$ can be extended to a 5-coloring of $G$
- Suppose that $|\{c(u) : u \in N(v)\}| = 5$
  - $N(v) = \{v_1, ..., v_5\}, c(v_i) = i, \forall i \in [5]$
The Five Color Theorem

- Every $v_1 \to v_3$ path $P \subseteq H$ separates $v_2$ from $v_4$
  - $C := vv_1 P v_3 v$
  - $v_2, v_4$ are in different faces of $C$
- $i, j \in [5]$: $H_{i,j}$ subgraph of $H$ induced by vertices with color $\{i, j\}$
  - Let $v_1 \in C_1$, a component of $H_{1,3}$
    - $v_3 \notin C_1 \Rightarrow$ change 1 to 3, 3 to 1 in $C_1$ gives a 5-coloring of $H$
    - Let $c(v) = 1$. This is a 5-coloring of $G$
Suppose that $P \subseteq C_1$

$v v_1 P v_3 v$ separates $v_2, v_4$ and $P \cap H_{2,4} = \emptyset \Rightarrow v_2, v_4$ are in different components of $H_{2,4}$

In the component of $v_2$ interchange 2,4

Let $c(v) = 2$. This gives a 5-coloring of $G$. 
Problem:
• How can the final exams at a university be scheduled so that no student has two exams at the same time?
Trees

Definition:
• A *tree* is a connected undirected graph with no simple circuits.

\[
\begin{align*}
&\text{tree} & \text{tree} & \exists \text{ circuits} & \text{not connected}
\end{align*}
\]
Characterizations

**Theorem:** Let $G = (V, E)$ be an undirected graph.

- $G$ is a tree iff there is a unique simple path between any 2 vertices.

  - $\Rightarrow$: $G$ is a tree $\Rightarrow$ $G$ is connected
    - $\forall u, v \in V, \exists$ a simple path $P$: $\{u, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k = v\}$
    - Suppose $\exists$ another simple path $P'$: $\{u, y_1\}, \{y_1, y_2\}, \ldots \{y_{l-1}, y_l = v\}$
    - Let $a := \min\{i \geq 1: x_i \in \{y_1, \ldots, y_{l-1}, y_l\}\}$ and $x_a = y_b$
    - $u, x_1, \ldots, x_a, y_{b-1}, \ldots, y_1, u$ is a simple circuit in $G$

  - $\Leftarrow$: $\forall u, v \in V, \exists$ path $u \rightarrow v \Rightarrow G$ is connected
    - Suppose there is a simple circuit $u, x_1, \ldots, x_k, u$ in $G$
      - $k \geq 2$
      - $u, x_1$ and $u, x_k, \ldots, x_1$ are two simple paths from $u \rightarrow v$
Characterizations

**Theorem:** Let $G = (V, E)$ be an undirected graph.

- $G$ is a tree iff $G$ is connected and $G - e$ is disconnected $\forall e \in E$.

  - $\Rightarrow$: $G$ is a tree $\Rightarrow G$ is connected
    - Let $e = \{u, v\} \in E$.
    - If $G - e$ is connected, then $\exists$ a simple path $u, x_1, \ldots, x_k, v$
    - Then $G$ has a circuit $u, x_1, \ldots, x_k, v, u$
    - $G$ has a simple circuit

  - $\Leftarrow$: need to show that $G$ has no simple circuits
    - Suppose there is a simple circuit $u, x_1, \ldots, x_k, u$ in $G$
      - $G - \{u, x_1\}$ is still connected, a contradiction.
Characterizations

**Theorem:** Let \( G = (V, E) \) be an undirected graph.

- \( G \) is a tree iff \( G \) is connected, has no simple circuits but \( G + e \) has a simple circuit \( \forall e \notin E \).

- \( \Rightarrow: G \) is a tree \( \Rightarrow G \) is connected
  
  - Let \( e = \{u, v\} \notin E \).
  
  - \( G \) is a tree \( \Rightarrow \exists \) a simple path \( u, x_1, ..., x_k, v \)
  
  - \( u, x_1, ..., x_k, v, u \) is a circuit in \( G + e \)
  
  - \( G + e \) has a simple circuit

- \( \Leftarrow: \) obvious, by definition.
Characterizations

**Theorem:** Let $G = (V, E)$ be a connected undirected graph.

- $V$ can be enumerated as $v_1, ..., v_n$ s.t. $G[v_1, ..., v_i]$ is connected
  - Pick $v_1 \in V$ arbitrarily.
  - $\exists v_2 \in V - v_1$ such that $v_1 v_2 \in E$. $G[v_1, v_2]$ is connected.
  - Suppose we have chosen $v_1, ..., v_i$
  - Need to choose $v_{i+1}$. Pick $u \in V - \{v_1, ..., v_i\}$
  - $G$ is connected $\Rightarrow \exists$ path $u \rightarrow v_1$
  - Let $v_{i+1}$ be the last vertex on $u \rightarrow v_1$ and in $G - G[v_1, ..., v_i]$
  - $v_{i+1}$ has a neighbor in $G[v_1, ..., v_i]$

**Theorem:** Let $G = (V, E)$ be a tree with $n$ vertices.

- We can enumerate $V = \{v_1, ..., v_n\}$ such that $v_i$ has a unique neighbor (no circuit) in $\{v_1, ..., v_{i-1}\}$ for every $i \in \{2, ..., n\}$. 
Characterizations

Theorem: Let $G = (V, E)$ be a connected undirected graph and $|V| = n$.

- $G$ is a tree iff $G$ has $n - 1$ edges.
  - $\Rightarrow$: $V = (v_1, \ldots, v_n)$ such that $v_i$ has a unique neighbor in $\{v_1, \ldots, v_{i-1}\}$ for every $i \geq 2$. Hence, there are exactly $n - 1$ edges.
  - $\Leftarrow$: $V = (v_1, \ldots, v_n)$ such that $G[v_1, \ldots, v_i]$ is connected $\forall i$
    - There are $n - 1$ edges.
      - $v_i$ is adjacent to exactly 1 vertex in $\{v_1, \ldots, v_{i-1}\}$, $\forall i$
      - Suppose $v_{i_1}, v_{i_2} \ldots, v_{i_k}, v_{i_1}$ is a simple circuit.
        - $v_{i_k}$ is adjacent to two vertices in $\{v_1, \ldots, v_{i_k-1}\}$
Forest

Definition: A forest is a graph whose components are trees.

A forest of 3 trees