Discrete Mathematics
Graph Theory

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Theorem: $G = (V = \{v_1, \ldots, v_n\}, E)$ is a graph with adjacency matrix $A$.

- directed or undirected edges, multiple edges and loops allowed
- The number of different paths of length $r$ from $v_i$ to $v_j$ is $[A^r]_{i,j}$
  - $r = 1$: # of paths $v_i \rightarrow v_j$ is $[A]_{i,j}$
  - Induction hypothesis: true for $r$
  - Need to prove for $r + 1$.
    - $\forall k \in [n], [A^r]_{i,k} = \#$ of paths $v_i \rightarrow v_k$ of length $r$
    - Let $P = v_i, \ldots, x, v_j$ be any path $v_i \rightarrow v_j$ of length $r + 1$
      - $N_k : \#$ of different $P$ with $x = v_k, k \in [n]$
      - $N_k = [A^r]_{i,k} \cdot [A]_{k,j}$ for every $k \in [n]$
      - $N_1 + \cdots + N_n = [A^{r+1}]_{i,j}$
Number of Paths

- # of paths $a \rightarrow d$ of length 4 is $[A^4]_{1,4} = 8$
- $ababbd, acabbd; abdbbd, acdbbd$
- $abaccd, acaccd; abdcd, acdcd$
Euler Paths and Circuits

**Definition:** Let $G = (V, E)$ be a (directed or undirected) graph.

- **Euler circuit:** a simple circuit that contains every edge of $G$.
- **Euler path:** a simple path that contains every edge of $G$. 

![Diagram of Euler circuits and paths in graphs](image-url)
Euler Circuits

**Theorem:** \( G = (V, E) \): connected (undirected) multigraph, \( |V| \geq 2 \).

- \( G \) has an Euler circuit iff \( 2 \mid \deg(u) \) for every \( u \in V \).
  
  \( \Rightarrow \): Let \( \{x_0, x_1\}, \ldots, \{x_{i-1}, x_i\}, \ldots, \{x_{n-1}, x_n\} \) be an Euler circuit, \( x_0 = x_n \)
  
  - Every \( x_i \) appears an even number of times in the circuit
    
    - This contributes a even number to \( \deg(x_i) \)
    
    - Every vertex \( x_i \) has an even degree
  
  \( \Leftarrow \): Let \( \{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\} \) be a longest simple path/circuit in \( G \)
  
  - If \( x_n \neq x_0 \), then \( \deg(x_n) \) must be odd. Hence, \( x_n = x_0 \)
  
  - If \( \exists e = \{y, x_i\} \in E \) such that \( e \) does not occur in the circuit
    
    - \( y, x_i, x_{i+1}, \ldots, x_n, x_0, \ldots, x_{i-1}, x_i \) is a longer simple path/circuit
Constructing Euler Circuits

Hierholzer's Algorithm:

• Input: $G = (V, E)$, a connected multigraph, $2|\text{deg}(x)|, \forall x \in V$
• Output: an Euler circuit
  • $\text{circuit} := \text{a circuit in } G$
  • $H := G - \text{circuit} - \text{isolated vertices}$
  • while $H$ has edges do
    • $\text{subcircuit} := \text{a circuit in } H$ that intersects $\text{circuit}$
    • $H := H - \text{subcircuit} - \text{isolated vertices}$
    • $\text{circuit} := \text{circuit} \cup \text{subcircuit}$
  • return $\text{circuit}$

Analysis: We can always find a subcircuit as long as $H \neq \emptyset$

• Every component of $H$ satisfies the condition of the Theorem.
• $\geq 2$ edges are added in each iteration. The worst case complexity is $O(|E|)$
Example

circuit = a, b, e, a
Example

\[
\text{circuit} = a, b, e, a
\]

\[
\text{subcircuit} = a, c, f, a
\]
Example

\[ \text{circuit} = a, b, e, a \]

\[ \text{subcircuit} = a, c, f, a \]
Example

\[
\text{circuit} = a, b, e, a, c, f, a
\]

\[H\]
Example

\[ \text{circuit} = a, b, e, a, c, f, a \]

\[ H \]

\[ \text{subcircuit} = c, d, e, f, b, c \]
Example

\[ \text{circuit} = a, b, e, a, c, f, a \]  
\[ \text{subcircuit} = c, d, e, f, b, c \]
Example

\[
\text{circuit} = a, b, e, a, c, d, e, f, b, c, f, a
\]
Constructing Euler Circuits

Fleury's Algorithm:
- Input: $G = (V, E)$, a connected multigraph, $2|\deg(x), \forall x \in V$
- Output: a Euler circuit
  - `circuit := ∅`
  - pick $v \in V$, find an edge $\{v, w\}$ which is not a cut edge
  - `circuit := circuit + {v, w}; H := G - {v, w} - isolated vertices`
  - while $H$ has edges do
    - $v := w$;
    - find an edge $vw$ which is not a cut edge
      - if there are only cut edges, choose one
    - `circuit := circuit + {v, w}; H := H - {v, w} - isolated vertices`
  - return `circuit`
Example
Example

Complexity: $O(|E|)$
Euler Paths

**Theorem:** $G = (V, E)$: connected multigraph, $|V| \geq 2$.

- Then $G$ has an Euler path (not Euler circuit) iff $G$ has exactly 2 vertices of odd degree.
  - $\Leftarrow$: Let $x, y \in V$ be of odd degree. Let $H := G + \{x, y\}$.
    - Every vertex has even degree in $H$
    - $H$ has a Euler circuit: $a \cdots x, y, \cdots a$
    - $y, \cdots, a, \cdots, x$ is a Euler path in $G$
  - $\Rightarrow$: $G$ has $k = 0, 2, 4, \ldots$ vertices of odd degree
    - $k = 0$: there is a Euler circuit
    - $k \geq 2$: suppose there is a Euler path (not circuit):
      - $\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\} (x_0 \neq x_n)$
      - The only possible vertices of odd degree are $x_0, x_n$
Constructing Euler Paths

Algorithm:

- **Input:** \( G = (V, E) \), a connected multigraph, \( x, y \) have odd degree
- **Output:** an Euler path
  - \( H := G + xy \)
  - find a Euler circuit using Hierholzer's or Fleury's algorithm
  - remove the edge \( xy \) from the circuit \( \Rightarrow \) an Euler path
Hamilton Paths and Circuits

**Definition:** Let $G = (V, E)$ be a graph.

- A simple path $x_0, x_1, ..., x_n$ is a **Hamilton path** if $V = \{x_0, x_1, ..., x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$.

- A simple circuit $x_0, x_1, ..., x_n, x_0$ is a **Hamilton circuit** if $x_0, x_1, ..., x_n$ is a Hamilton path.
Sufficient Conditions

Ore’s Theorem: Let $G = (V, E)$ be a simple graph

- "$|V| = n \geq 3$ and $\text{deg}(u) + \text{deg}(v) \geq n$ whenever $\{u, v\} \notin E$" $\Rightarrow G$ has a Hamilton circuit.

Dirac’s Theorem: Let $G = (V, E)$ be a simple graph

- If $|V| = n \geq 3$ and $\text{deg}(u) \geq n/2$ for every $u \in V$, then $G$ has a Hamilton circuit.
Ore’s Theorem

Ore’s Theorem: Let $G = (V, E)$ be a simple graph

• $|V| = n \geq 3$ and $\text{deg}(u) + \text{deg}(v) \geq n$ whenever $\{u, v\} \notin E$ \ \Rightarrow \ G$ has a Hamilton circuit.

• Operations on $G$:
  • $H := G$;
  • while $H$ does not contain a Hamilton circuit do
    • $G' = H$;
    • pick $u, v \in V$ s.t. $\{u, v\}$ is not an edge of $G'$
    • $H := G' + \{u, v\}$;
    • $x := u; y := v$;
  • Let $x = u_1, u_2, \ldots, u_{n-1}, y = u_n$ be a Hamilton circuit in $H$. 
Ore’s Theorem

Let $A := \{i: 2 \leq i \leq n, \{x, u_i\} \in E\}$

Let $B := \{i: 2 \leq i \leq n, \{y, u_{i-1}\} \in E\}$

$\{x, y\} \notin E \Rightarrow |A| + |B| \geq n \Rightarrow \exists i \in \{2, ..., n\}$ s. t. $i \in A \cap B$

- $\{x, u_i\} \in E$ and $\{y, u_{i-1}\} \in E$

- $x = u_1, u_2, ..., u_{n-1}, y = u_n$ is a Hamilton circuit in $H$
  - $x = u_1, ..., u_{i-1}, y = u_n, u_{n-1}, ..., u_i, x = u_1$ is a Hamilton circuit in $G'$
  - $G'$ does not have Hamilton circuit $\rightarrow \leftarrow$
Application

Travel Salesperson Problem:

• There is a list of cities
• The distances between each pair of cities are given
• Find a route that *visits each city exactly once* and *returns to the origin city* such that the *total distance is minimum*.
  • *NP-hard* in combinatorial optimization
    • operations research; theoretical computer science

\[ \text{ABCDA} \]