Discrete Mathematics

Graph Theory

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Seven Bridges of Königsberg

The Problem:

• Is it possible to travel all seven bridges without repetition?
• $V = \{A, B, C, D\}$; $E = \{1, 2, 3, 4, 5, 6, 7\}$
  
• $1 = \{A, B\}, 2 = \{B, C\}, ...$
Definition: \textit{undirected graph} \( G = (V, E) \)

- \( V = \{v_1, \ldots, v_n\} \), a nonempty set of \textit{vertices}
- \( E \): a set of \textit{edges}, \( \forall e \in E, e = \{v_i, v_j\} \) for some \( v_i, v_j \in V \)
  - \( v_i, v_j \) are \textit{endpoints} of an edge \( \{v_i, v_j\} \); \( \{v_i, v_j\} \) \textit{connects} \( v_i \) and \( v_j \)
  - \textit{multiple edges}: \( e \in E \) appears \( > 1 \) times in \( E \)
  - \textit{loops}: an edge of the form \( \{v_i, v_i\} \)
- \textit{simple graph}: no multiple edges; no loops
- \textit{multigraph}: multiple edges allowed; loops not allowed
- \textit{pseudograph}: multiple edges and loops are both allowed
Examples

Simple graph

Vertex set: $V = \{1, 2, 3, 4, 5\}$

Edge set: $E = \{12, 23, 34, 14, 45\}$

$\{4, 5\}$ is an edge of the simple graph

$4, 5$ are endpoints of the edge $\{4, 5\}$

$\{4, 5\}$ connects 4 and 5.

$\{3, 4\}$ is a multiple edge

There is a loop connecting 1 to itself

Multigraph

Pseudograph
Definition: **directed graph** $G = (V, E)$

- each edge is an ordered pair (e.g., $(v_i, v_j)$).
  - the edges are called **directed edges** or **arcs**
  - $(v_i, v_j)$ **starts** at $v_i$ and **ends** at $v_j$
  - $(v_i, v_j)$ has **multiplicity** $m$ if it appears $m$ times in $E$
- **simple directed graphs**: no multiple directed edges; no loops
- **directed multigraphs**: directed multiple edges and loops allowed
- **mixed graphs**: directed and undirected edges both allowed
Examples

Simple directed graph

Directed multigraph

Mixed graph

• Vertices: $V = \{1, 2, 3, 4, 5\}$
• Edges: $E = \{(1, 2), (2, 3), (3, 4), (4, 1), (5, 4)\}$
• $(5, 4)$ is an arc or directed edge
• $(5, 4)$ starts at 5 and ends at 4
• $(3, 4)$ is a directed multiple edge
• There is a loop connecting 1 to itself
Decide Types of Graphs

Remarks: Graph is a general term to describe graphs of all types

<table>
<thead>
<tr>
<th>Type</th>
<th>Edges</th>
<th>Multiple Edges</th>
<th>Loops</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple graph</td>
<td>undirected</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>multigraph</td>
<td>undirected</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>pseudograph</td>
<td>undirected</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Simple directed graph</td>
<td>directed</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Directed multigraph</td>
<td>directed</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Mixed graph</td>
<td>both</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Three Key Questions:
- Are the edges undirected or directed (or both)?
- Are there any multiple (directed) edges?
- Are there any loops?
Social Networks

Collaboration Graphs: (simple graph)
- Vertices: people
- Edges: 2 people are connected by an undirected edge when the people have collaborated
  - Academic collaboration graph (Erdös number)
    - Vertices: researchers
    - Edges: 2 people are connected by an edge if they have jointly published a paper.
Precedence Graphs: (directed simple graph)

- Vertices: statements
- Edges: there is an arc \((u, v)\) if \(v\) cannot be executed before \(u\)
Communication Networks

Call Graphs: (directed multigraphs)
- Vertices: telephone numbers
- Edges: there is an arc \((u, v)\) if \(u\) called \(v\)
- AT&T experiment: calls during 20 days (290 million vertices and 4 billion edges)
Transportation Network

**Airline Routes:** (directed multigraphs)
- Vertices: airports
- Edges: there is an arc \((u, v)\) if there is a flight from \(u\) to \(v\)
Degree (Undirected Graphs)

**Definition:** $G = (V, E)$ – an undirected graph

- $u, v \in V$ are **adjacent** (or **neighbors**) if $\{u, v\} \in E$.
- an edge $e = \{u, v\}$ is **incident** with $u, v$, or **connect** $u, v$
  - the **neighborhood** of $v$ in $G$ is $N(v) = \{u \in V: \{u, v\} \in E\}$
  - the **neighborhood** of $A \subseteq V$ is $N(A) = \bigcup_{v \in A} N(v)$
  - the **degree** of $v \in V$ in $G$, denoted by $\deg(v)$, is the number of edges incident with $v$
    - every loop $\{v\}$ contributes 2 to $\deg(v)$
    - $v$ is **isolated** if $\deg(v) = 0$
    - $v$ is **pendant** if $\deg(v) = 1$
Example

• 4 and 5 are adjacent in the graph
• The edge \{4,5\} is incident with 4 and 5
• \(N(4) = \{1,3,5\}\)
• \(N(\{1,4\}) = \{1,2,3,4,5\}\)
• \(\deg(1) = 4, \deg(2) = 2, \deg(3) = 3, \deg(4) = 4, \deg(5) = 1\)
• 6 is isolated
• 5 is pendant
Handshaking Theorem

**Theorem:** Let $G = (V, E)$ be an undirected graph with $m$ edges. Then

- $2m = \sum_{v \in V} \deg(v)$ and $\# \{ v \in V : 2 \nmid \deg(v) \}$ is even.
- The endpoints of any edge $e \in E$ contribute 2 to $\sum_{v \in V} \deg(v)$
  - $e = \{v_i, v_j\}$: $e$ contributes 1 to $\deg(v_i)$ and 1 to $\deg(v_j)$
  - $e = \{v_i\}$: $e$ contributes 2 to $\deg(v_i)$
- $m$ edges contribute $2m$ to $\sum_{v \in V} \deg(v)$
- $\sum_{v \in V} \deg(v) = \sum_{v \in V : 2 \mid \deg(v)} \deg(v) + \sum_{v \in V : 2 \nmid \deg(v)} \deg(v)$
  - $2 \mid \sum_{v \in V} \deg(v)$; $2 \mid \sum_{v \in V : 2 \mid \deg(v)} \deg(v)$ \Rightarrow
  - $2 \mid \sum_{v \in V : 2 \nmid \deg(v)} \deg(v) \Rightarrow \# \{ v \in V : 2 \nmid \deg(v) \}$ is even
Degree (Directed Graphs)

Definitions: $G = (V, E)$-directed graph

- $(u, v) \in E$: $u$ is adjacent to $v$ and $v$ is adjacent from $u$.
  - $u$: initial vertex of $(u, v)$
  - $v$: terminal vertex or end vertex of $(u, v)$
  - $u = v$: $u$ is the initial vertex and the terminal vertex
- in-degree $\deg^-(v)$: # of edges where $v$ is the terminal vertex
- out-degree $\deg^+(v)$: # of edges where $v$ is the initial vertex
  - $u = v$: the loop contributes 1 to $\deg^-(v)$ and $\deg^+(v)$
- 5 is adjacent to 4
- 4 is adjacent from 5
- 5 is the initial vertex of (5,4)
- 4 is the terminal vertex of (5,4)
- 1 is the initial and terminal vertex of the loop of 1
- \( \deg^-(1) = 2, \deg^+(1) = 2; \deg^-(4) = 3; \deg^+(4) = 1 \)
Theorem: Let $G = (V, E)$ be a graph with directed edges. Then
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$ 

- Every arc $e \in E$ contributes 1 to $\sum_{v \in V} \deg^-(v)$
  - $e = (v_i, v_j)$ contributes 1 to $\deg^-(v_i)$
  - $e = (v_i, v_i)$ contributes 1 to $\deg^-(v_i)$
- Every arc $e \in E$ contributes 1 to $\sum_{v \in V} \deg^+(v)$
  - $e = (v_i, v_j)$ contributes 1 to $\deg^+(v_j)$
  - $e = (v_i, v_i)$ contributes 1 to $\deg^+(v_i)$
**Special Simple Graphs**

**Complete Graphs:** $K_n$ - complete graph on $n$ vertices

- $V = \{v_1, v_2, \ldots, v_n\}$;
- $E = \{\{v_i, v_j\}: 1 \leq i \neq j \leq n\}$
Special Simple Graphs

**Cycle:** $C_n$-cycle on $n$ vertices

- $V = \{v_1, v_2, ..., v_n\}$
- $E = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$
Special Simple Graphs

Wheels: $W_n$-wheel on $n$ vertices

- $V = \{v_0, v_1, v_2, \ldots, v_n\}$
- $E = \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\} \cup \{\{v_0, v_1\}, \ldots, \{v_0, v_n\}\}$
**Special Simple Graphs**

**n-Cubes:** $Q_n$ - $n$-dimensional hypercube, or $n$-cube,

- $V = \{0,1\}^n$
- $E = \{\{u, v\}: d(u, v) = 1\}$
  - $d(u, v) = |\{i \in [n]: u_i \neq v_i\}|$
Special Simple Graphs

Bipartite graphs

- \( V = V_1 \cup V_2 \), where \( V_1 \cap V_2 = \emptyset \)
- \( E \subseteq \{\{u_1, u_2\}: u_1 \in V_1, u_2 \in V_2\} = V_1 \times V_2 \)
  - \((V_1, V_2)\) is a **bipartition** of the vertex set \( V \) of \( G = (V, E) \).
Special Simple Graphs

Complete Bipartite Graphs $K_{m,n}$

- $V = \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$
- $E = \{\{x_i, y_j\}: i \in [m], j \in [n]\}$
Theorem: $G = (V, E)$ is a simple graph

- $G$ is bipartite iff $\exists f : V \to \{1, 2\}$ s.t. "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)"
  
  - $\Rightarrow$: $G = (A \cup B, E)$, where $A \cap B = \emptyset$
    - $f : V \to \{1, 2\} \ x \mapsto 1$ if $x \in A; x \mapsto 2$ if $x \in B$
    - $\{x, y\} \in E \Rightarrow x \in A, y \in B$ or $x \in B, y \in A \Rightarrow f(x) \neq f(y)$

- $\Leftarrow$: $f : V \to \{1, 2\}$ is a map s.t. "$\{x, y\} \in E \Rightarrow f(x) \neq f(y)"
  
  - $A = f^{-1}(1), B = f^{-1}(2) : V = A \cup B, A \cap B = \emptyset$
  - $\{x, y\} \in E \Rightarrow f(x) \neq f(y) \Rightarrow x \in A, y \in B$ or $x \in B, y \in A$
  - $G$ must be a bipartite graph.
Matching

**Definition:** $G = (V, E)$ is a simple graph

- $M \subseteq E$ is called a **matching** if $e \cap e' = \emptyset$ for every $e, e' \in M$.
- $v \in V$ is **matched** in $M$ if $\exists e \in M$ such that $v \in e$
- otherwise, $v$ is **not matched**

- **maximum matching:** a matching with largest number of edges.

- In a bipartite graph $G = (A \cup B, E)$, we say that $M \subseteq E$ is a **complete matching** from $A$ to $B$ if every $u \in A$ is matched.
Example

- $V = \{a, b, c, d, u, v, w, x, y\}$; $A = \{a, b, c, d\}$ to $B = \{u, v, w, x, y\}$
- $E = \{au, aw, bv, bx, cu, cv, dw, dy\}$
- $M = \{au, bv\}$ is a matching
- $a, b, u, v$ are matched in $M$
- $c, d, x, y$ are not matched in $M$
- $M$ is not a maximum matching
- $M' = \{aw, bv, cu, dy\}$ is a maximum matching
- $M'$ is a complete matching from $A$ to $B$
Marriages on an Island

Example: (Marriages on an Island)

- There are $m$ men $X = \{x_1, \ldots, x_m\}$ and $n$ women $Y = \{y_1, \ldots, y_n\}$
- $G = (X \cup Y, E = \{\{x_i, y_j\}: x_i \text{ and } y_j \text{ can get married}\}$
- What is the largest number of couples that can be formed?

Hall’s Marriage Theorem

- The bipartitie graph $G = (X \cup Y, E)$ has a complete matching from $X$ to $Y$ iff $|N(A)| \geq |A|$ for any $A \subseteq X$. 