Discrete Mathematics
Combinatorial Counting

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Fibonacci Numbers

Problem:

• A young pair of rabbits is placed on an island.
• A pair of rabbits does not breed until they are 2 months old.
• Each pair of 2-month old rabbits produce 1 pair every month.
  • $f_n$: # of pairs at the end of the $n$th month ($n = 1, 2, ...$)
    • $f_1 = 1, f_2 = 1, f_3 = f_2 + f_1 = 2, f_4 = f_3 + f_2 = 3, ...$
    • $f_n = f_{n-1} + f_{n-2}$ for every $n \geq 3$
  • $f_n =$?
The Tower of Hanoi

Problem:

- Only 1 disk can be moved from one peg to another every time.
- A larger disk cannot be placed on top of a smaller disk.
- Move all the disks from peg 1 to peg 2.
  - $H_n$: the smallest number of moves ($n$ disks).
    - $H_1 = 1$, $H_2 = 3$, $H_n = 2H_{n-1} + 1$ for $n \geq 2$
    - $H_n =$?
Strings

Problem:

- \( a_n \): # of \( n \)-bit strings that do not contain “00”. \((n = 1, 2, \ldots)\)
  - \( a_1 = 2, a_2 = 3, a_3 = 5 \)
  - \( a_n = a_{n-1} + a_{n-2} \) for \( n \geq 3 \)
    - \( x_1 \ldots x_{n-1}0: a_{n-2} \)
    - \( x_1 \ldots x_{n-1}1: a_{n-1} \)

- \( a_n \): # of \( n \)-digit integers that contain an even number of 0s \( n \geq 1 \)
  - \( a_1 = 9, a_2 = 82 \)
  - \( a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1} \)
    - \( x_1 \ldots x_{n-1}0: 10^{n-1} - a_{n-1} \)
    - \( x_1 \ldots x_{n-1}(\neq 0): 9a_{n-1} \)
Linear Homogeneous Recurrence Relations

Definition:

• A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

\[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \]

where \( c_1, c_2, \ldots, c_k \) are real numbers, and \( c_k \neq 0 \).

• **linear**: RHS is a linear combination of \( a_1, a_2, \ldots, a_{n-1} \).

• **homogeneous**: every term is a multiple of some \( a_j \).

• **degree k**: depend on \( k \) previous terms

• **constant coefficients**: \( c_1, \ldots, c_k \) do not depend on \( n \)

Example:

• \( f_n = f_{n-1} + f_{n-2}, n \geq 3 \)

• \( H_n = 2H_{n-1} + 1, n \geq 2 \)
Theorem: $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$

• **characteristic equation**:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

• **characteristic roots**: solutions of this equation
Solving L.H.R.R

**Theorem:** $c_1, c_2 \in \mathbb{R}$ and $r^2 - c_1 r - c_2 = 0$ has two roots $r_1 \neq r_2$.

- The solutions of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where $\alpha_1$ and $\alpha_2$ are constants and $n \geq 0$

- $\text{RHS} = c_1 (\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2 (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$

  $= \alpha_1 (c_1 r_1^{n-1} + c_2 r_1^{n-2}) + \alpha_2 (c_1 r_2^{n-1} + c_2 r_2^{n-2})$

  $= \alpha_1 r_1^{n-2} (c_1 r_1 + c_2) + \alpha_2 r_2^{n-2} (c_1 r_2 + c_2)$

  $= \alpha_1 r_1^n + \alpha_2 r_2^n$

  $= a_n$
Solving L.H.R.R

**Theorem:** $c_1, c_2 \in \mathbb{R}$ and $r^2 - c_1 r - c_2 = 0$ has two roots $r_1 \neq r_2$.

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- Let $\{b_n\}$ be any solution of the LHRR

- Let $\alpha_1 + \alpha_2 = b_0$; $\alpha_1 r_1 + \alpha_2 r_2 = b_1$

  - $\alpha_1 = \frac{b_1 - r_1 b_0}{r_2 - r_1}$; $\alpha_2 = \frac{b_1 - r_2 b_0}{r_1 - r_2}$

- $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ with the above $\alpha_1, \alpha_2$ is a solution

- $\{a_n\}$ and $\{b_n\}$ have the same initial condition
Example

• Solving $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$
  • $r^2 - r - 2 = 0 \Rightarrow r_1 = -1, r_2 = 2$
  • $a_n = \alpha_1 (-1)^n + \alpha_2 2^n; a_0 = 2, a_1 = 7 \Rightarrow \alpha_1 = -1, \alpha_2 = 3$

• Explicit formula of Fibonacci numbers
  • $r^2 - r - 1 = 0 \Rightarrow r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
  • $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
  • $a_1 = 1, a_2 = 1 \Rightarrow \alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = -\frac{1}{\sqrt{5}}$
Theorem $c_1, c_2 \in \mathbb{R}, c_2 \neq 0$ and $r^2 - c_1 r - c_2 = 0$ has a unique root $r_0$.

- The solutions of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$
  where $\alpha_1, \alpha_2$ are constants and $n \geq 0$

- We must have that $r_0 = c_1 / 2$, i.e., $c_1 = 2r_0$

- RHS = $c_1 (\alpha_1 r_0^{n-1} + \alpha_2 (n - 1) r_0^{n-1}) + c_2 (\alpha_1 r_0^{n-2} + \alpha_2 (n - 2) r_0^{n-2})$
  \[= \alpha_1 r_0^{n-2} (c_1 r_0 + c_2) + \alpha_2 (n - 2) r_0^{n-2} (c_1 r_0 + c_2) + c_1 \alpha_2 r_0^{n-1}\]
  \[= \alpha_1 r_0^n + \alpha_2 (n - 2) r_0^n + c_1 \alpha_2 r_0^{n-1}\]
  \[= \alpha_1 r_0^n + \alpha_2 r_0^n\]
Solving L.H.R.R

Theorem $c_1, c_2 \in \mathbb{R}, c_2 \neq 0$ and $r^2 - c_1 r - c_2 = 0$ has a unique root $r_0$.

- The solutions of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$

  where $\alpha_1, \alpha_2$ are constants and $n \geq 0$

- Let $\{b_n\}$ be any solution of the LHRR
- Let $b_0 = \alpha_1; b_1 = \alpha_1 r_0 + \alpha_2 r_0$
  - $\alpha_1 = b_0; \alpha_2 = (b_1 - b_0 r_0)/r_0$
  - $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ is a solution
- $\{a_n\}$ and $\{b_n\}$ satisfy the same initial condition
Example

- Solving \(a_n = 6a_{n-1} - 9a_{n-2}\) with \(a_0 = 1\) and \(a_1 = 6\)
  - \(r^2 - 6r + 9 = 0 \Rightarrow r_0 = 3\)
  - \(a_n = \alpha_1 3^n + \alpha_2 n 3^n\)
  - \(a_0 = 1, a_1 = 6 \Rightarrow \alpha_1 = \alpha_2 = 1\)
  - \(a_n = 3^n + n3^n = 3^n(n + 1)\)
Solving L.H.R.R

Theorem: (all roots are distinct)

• \(c_1, c_2, \ldots, c_k \in \mathbb{R}\)
• \(r^k - c_1 r^{k-1} - \cdots - c_k = 0\) has \(k\) distinct roots \(r_1, r_2, \ldots, r_k\).
• Then the solutions of \(a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}\) are
  \[a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n\]
  where \(\alpha_1, \alpha_2, \ldots, \alpha_k\) are constants and \(n \geq 0\)
• \(\text{RHS} = \sum_{i=1}^{k} c_i a_{n-i} = \sum_{i=1}^{k} c_i \sum_{j=1}^{k} \alpha_j r_j^{n-i}\)
  \[= \sum_{j=1}^{k} \alpha_j \sum_{i=1}^{k} c_i r_j^{n-i}\]
  \[= \sum_{j=1}^{k} \alpha_j r_j^{n-k} \sum_{i=1}^{k} c_i r_j^{k-i}\]
  \[= \sum_{j=1}^{k} \alpha_j r_j^{n-k} \cdot r_j^k\]
  \[= \sum_{j=1}^{k} \alpha_j r_j^n\]
Solving L.H.R.R

Theorem: (all roots are distinct)

- \( c_1, c_2, \ldots, c_k \in \mathbb{R} \)
- \( r^k - c_1 r^{k-1} - \cdots - c_k = 0 \) has \( k \) distinct roots \( r_1, r_2, \ldots, r_k \).
- Then the solutions of \( a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} \) are
  \[
  a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n
  \]
  where \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are constants and \( n \geq 0 \)
- Let \( \{b_n\} \) be any solution of the LHRR. Solve

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
r_1 & r_2 & \cdots & r_k \\
\vdots & \vdots & \cdots & \vdots \\
r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{pmatrix}
= 
\begin{pmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{k-1}
\end{pmatrix}
\]
Solving L.H.R.R

**Theorem: (all roots are distinct)**

- $c_1, c_2, \ldots, c_k \in \mathbb{R}$
- $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has $k$ distinct roots $r_1, r_2, \ldots, r_k$.
- Then the solutions of $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ are
  $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$
  where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants and $n \geq 0$
- $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ is a solution
- $\{a_n\}$ and $\{b_n\}$ satisfy the same initial conditions
- $a_n = b_n$ for every $n$
Example

Solving $a_n = 6 \ a_{n-1} - 11 \ a_{n-2} + 6 \ a_{n-3}$ with $a_0 = 2, a_1 = 5, a_2 = 15$

- $r^3 - 6r^2 + 11r - 6 = 0 \Rightarrow r_1 = 1, r_2 = 2, r_3 = 3$
- $a_n = \alpha_1 \ 1^n + \alpha_2 \ 2^n + \alpha_3 \ 3^n = \alpha_1 + \alpha_2 \ 2^n + \alpha_3 \ 3^n$
- $a_0 = 2, a_1 = 5, a_2 = 15 \Rightarrow \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2$
- $a_n = 1 - 2^n + 2 \cdot 3^n$
Solving L.H.R.R

Theorem: (the roots are not distinct)

- \(c_1, c_2, \ldots, c_k \in \mathbb{R}\).
- \(r^k - c_1 r^{k-1} - \cdots - c_k = 0\) has \(t\) distinct roots \(r_1, r_2, \ldots, r_t\) with multiplicities \(m_1, m_2, \ldots, m_t\),
  - \(t \leq k\), \(m_1, \ldots, m_t \geq 1\) and \(m_1 + m_2 + \cdots + m_t = k\).
- Then the solutions of \(a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}\) are
  \[
  a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) \cdot r_1^n + \\
  (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) \cdot r_2^n + \cdots + \\
  (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) \cdot r_t^n
  \]
  where \(\alpha_{i,j}\)'s are constants and \(n \geq 0\).