Linear Control Systems

- Introduction
- Control constraints
- Parameterization of control functions

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Linear Control System

We use the notation

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \]

to denote linear control systems.

- \( x : \mathbb{R} \rightarrow \mathbb{R}^{n_x} \) denotes the state trajectory
- \( u : \mathbb{R} \rightarrow \mathbb{R}^{n_u} \) denotes the control input
- Functions \( A : \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x} \), \( B : \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_u} \) and \( b : \mathbb{R} \rightarrow \mathbb{R}^{n_x} \) given and integrable.

Example: double integrator

The double integrator:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= u(t)
\end{align*}
\]

is obtained for

\[
A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad b = 0.
\]

- \( x_2(t) \) (="velocity") corresponds to integral over \( u \) (="force")
- \( x_1(t) \) (="position") corresponds to integral over \( x_1 \).

Example: DC-motor

- source with voltage \( V_s(t) \)
- motor with resistance \( R \) and self-inductance \( L \)
- load (inertia) at the motor is denoted by \( J \)
Example: DC-motor

- Torque $T(t)$ assumed to be proportional to current $i(t)$, $T(t) = ki(t)$ ($k$ = given constant)
- Electrical power equals mechanical power, $v_{\text{motor}}(t)i(t) = T(t)\omega(t) \Rightarrow v_{\text{motor}}(t) = k\omega(t)$
- Kirchhoff’s voltage law: $V_s(t) = Ri(t) + L\frac{d}{dt}i(t) + k\omega(t)$
- Sum of inertial and friction torque: $T(t) = J\dot{\omega}(t) + \gamma\omega(t)$.

Substituting all equation yields a linear control system in standard form:

$$\frac{d}{dt}\begin{pmatrix} i(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{k}{L} \\ \frac{R}{J} & \frac{\gamma}{J} \end{pmatrix}\begin{pmatrix} i(t) \\ \omega(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_s(t).$$

States: current $i(t)$ and angular velocity $\omega(t)$
Input: voltage $V_s(t)$

Existence and uniqueness of state trajectories

If $u$ is integrable, the differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \quad \text{with} \quad x(0) = x_0$$

has for every initial value $x_0$ a unique solution,

$$x(t) = G(t,0)x_0 + \int_0^t G(t,\tau)(B(\tau)u(\tau) + b(\tau)) \, d\tau.$$

Recall: $G$ denotes fundamental solution, $\frac{d}{dt}G(t,\tau) = A(t)G(t,\tau)$, $G(\tau,\tau) = I$.

Control constraints

In most if not all applications there are physical limitations on the control input.

For given control constraint set $U \subseteq \mathbb{R}^{n_u}$ require

$$u(t) \in U$$

for all $t \in \mathbb{R}$. 
Control constraints

- In practice: $\mathcal{U}$ is often a simple box
  
  $$\mathcal{U} = [\underline{u}, \overline{u}] = \{ u \in \mathbb{R}^{n_u} \mid \forall i \in \{1, \ldots, n_u\}, \; \underline{u}_i \leq u_i \leq \overline{u}_i \}$$

  with lower and upper bounds $\underline{u} \leq \overline{u} \in \mathbb{R}^{n_u}$.

- Assuming that $\mathcal{U}$ is closed and convex is hardly ever restrictive.

- One exception are integer-valued controls (next slide).

Example: integer-valued constraints

If we fill a bottle with water, the amount $x(t)$ of water in the bottle satisfies

$$\dot{x}(t) = b \, u(t).$$

Here, $b \in \mathbb{R}_+$ denotes the maximum flow rate of the water, e.g., in liter per second.

- if valve is either open, $u(t) = 1$, or closed”, $u(t) = 0$, we set

  $$\mathcal{U} = \{0, 1\} \subseteq \mathbb{Z}.$$ 

Control saturation function

We use the notation $\text{sat}: \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$ to denote the control saturation function for given closed and convex set $\mathcal{U}$,

$$\forall u \in \mathbb{R}^{n_u}, \quad \text{sat}(u) = \arg\min_{v \in \mathcal{U}} \|u - v\|_2.$$ 

- frequently employed in control theory and algorithms

- If $\mathcal{U}$ is an interval, $\text{sat}$ can be evaluated by implementing a simple “if-else” switch

  (takes a few nano-seconds only).

Example: interval bounds

If $\mathcal{U} = [\underline{u}, \overline{u}] \subseteq \mathbb{R}$ is a simple interval,

$$\text{sat}(u) = \begin{cases} 
  u & \text{if } u \in [\underline{u}, \overline{u}] \\
  \underline{u} & \text{if } u < \underline{u} \\
  \overline{u} & \text{if } u > \overline{u}.
\end{cases}$$

For interval boxes $\mathcal{U}$ with $n_u > 1$ the function $\text{sat}$ can be evaluated componentwise.
Properties of the control saturation function

- If a control function \( u(t) \in U \) satisfies the control constraints, then we have \( \text{sat}(u(t)) = u(t) \).
- The function sat satisfies the control constraints \( \text{sat}(u(t)) \in U \) for all \( u \) by construction.
- The function \( \text{sat} \) is Lipschitz continuous with Lipschitz constant 1, i.e., we have
  \[
  \forall u_1, u_2 \in \mathbb{R}^{n_u}, \quad \| \text{sat}(u_1) - \text{sat}(u_2) \|_2 \leq \| u_1 - u_2 \|_2 .
  \]

Parameterization of control functions

Affine parameterizations of the control function can be written in the form
\[
u(t) = \sum_{i=0}^{M} a_i \varphi_i(t).
\]

- The functions \( \varphi_0, \varphi_1, \ldots, \varphi_M : \mathbb{R} \to \mathbb{R}^{n_u} \) are given basis functions.
- The scalars \( a_1, a_2, \ldots, a_M \in \mathbb{R} \) are the control parameterization coefficients.

Example: step response

For \( M = 1, n_u = 1 \) consider the basis functions
\[
\varphi_0(t) = \begin{cases} 1 & \text{if } t < 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_1(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]
models a single control switch parameterization. For the system
\[
x(t) = Ax(t) + Bu(t) \quad \text{and} \quad x(0) = 0
\]
(with \( A \) being invertible) the associated step response is
\[
x(t) = \int_0^t e^{A(t-\tau)} B a_1 \, d\tau = \left[ \left( e^{At} - I \right) A^{-1} B \right] a_1.
\]
Example: piecewise constant control functions

Piecewise constant control parameterization over \( M + 1 \) stages,

\[ h := \frac{T}{M+1}, \]

use

\[ \forall i \in \{0, \ldots, M\} \quad \varphi_i(t) = \begin{cases} 1 & \text{if } ih \leq t \leq (i+1)h \\ 0 & \text{otherwise}. \end{cases} \]

For LTI system \( \dot{x}(t) = Ax(t) + Bu(t) \) with \( x(0) = 0 \) and \( n_u = 1 \), the corresponding states \( \chi_k = x(kh) \) at the mesh points satisfy

\[ \chi_{k+1} = A\chi_k + B a_k \quad \text{with} \quad \chi_0 = 0, \]

with \( A = e^{Ah} \) and \( B = \int_0^h e^{A(t-\tau)} B \, d\tau. \)

Frequency response

For a given frequency \( \omega \) define

\[ \varphi_0(t) = \cos(\omega t) = \text{Re} \left( e^{i\omega t} \right) \quad \text{and} \quad \varphi_1(t) = \sin(\omega t) = \text{Im} \left( e^{i\omega t} \right) \]

with \( i = \sqrt{-1}. \) For LTI \( \dot{x}(t) = Ax(t) + Bu(t) \) with \( n_u = 1 \) the associated function \( \Phi(t) = \Phi_0(t) + \Phi_1(t)i \) can be written in the form

\[ \Phi(t) = \int_0^t e^{A(t-\tau)} B e^{i\omega \tau} \, d\tau = [A - i\omega I]^{-1} \left( e^{At} - e^{i\omega t} I \right) B \]

assuming that \( A \) has no eigenvalues that are equal to \( i\omega. \)

(more explicit expressions for real and imaginary part as well as a general Fourier analysis is part of Homework 6)