Linear Discrete-Time Systems

- Introduction
- Construction of linear discrete-time systems
- Condensing
- Matrix logarithms

Existence and Uniqueness

- There exists a unique sequence \( \chi_0, \chi_1, \ldots \in \mathbb{R}^{n_x} \), which satisfies the recursion for all \( k \geq 0 \).
- If the matrices \( A_k \) are invertible for all \( k \in \mathbb{Z} \),
  \[ \chi_k = A_k^{-1} (\chi_{k+1} - b_k) \quad \text{with} \quad \chi_0 = x_0, \]
  for all indices \( k \in \mathbb{Z} \), i.e., \( \chi_k \) exists for all integers \( k \in \mathbb{Z} \).

Example: Poincare maps

- Consider the periodic system
  \[ \dot{x}(t) = A(t)x(t) \quad \text{with} \quad x(0) = x_0 \]
  and \( A(t + T) = A(t) \) for a \( T > 0 \) and all \( t \in \mathbb{R} \); define
  \[ \xi_k = x(kT). \]
- Associated discrete-time recursion:
  \[ \forall i \in \mathbb{Z}, \quad \xi_{i+1} = A \xi_i \quad \text{with} \quad \xi_0 = x_0, \]
  where \( A = G(T, 0) \) is the monodromy matrix.
- The state recursion map \( \xi \rightarrow A \xi \) is called Poincare map.
Discretization of continuous-time systems

Recall: solution of linear continuous-time systems

\[
\dot{x}(t) = A(t)x(t) + b(t)
\]

can be written as

\[
x(t) = G(t,0)x_0 + \int_0^t G(t,\tau)b(\tau) \, d\tau .
\]

Discretization of continuous-time systems

- Define \( A_k = G(t_{k+1}, t_k) \) and \( b_k = \int_{t_k}^{t_{k+1}} G(t_k+1, \tau)b(\tau) \, d\tau \).
- State recursion at discrete-time points:

\[
x(t_{k+1}) = G(t_{k+1}, 0)x_0 + \int_0^{t_{k+1}} G(t_{k+1}, \tau)b(\tau) \, d\tau = G(t_{k+1}, t_k) \left[ G(t_k, 0)x_0 + \int_0^{t_k} G(t_k, \tau)b(\tau) \, d\tau \right] + \int_{t_k}^{t_{k+1}} G(t_{k+1}, \tau)b(\tau) \, d\tau = A_kx(t_k) + b_k .
\]

Remark: the matrices \( A_k \) are invertible.

Reducing the time resolution

We can reduce the resolution of a linear discrete time system by setting

\[
\forall k \in \mathbb{Z}, \quad \zeta_k = \chi_{mk}
\]

with a given \( m \in \{1, 2, 3, \ldots \} \).

- We store only every \( m \)-th element of the original state sequence \( \chi \).
- Advantage: saves memory.
- Disadvantage: time resolution of the state-sequence is worse.

Condensing

Consider the recursions

\[
E_{k,j+1} = A_{mk+j} \cdot E_{k,j} \quad \text{with} \quad E_{k,0} = I
\]

\[
e_{k,j+1} = A_{mk+j}e_{k,j} + b_{mk+j} \quad \text{with} \quad e_{k,j} = 0
\]

for all \( k \in \mathbb{Z} \) and all \( j \in \{0, \ldots, m - 1\} \). Sequence \( \zeta \) satisfies

\[
\forall k \in \mathbb{Z}, \quad \zeta_{k+1} = E_{k,m} \zeta_k + e_{k,m} \quad \text{with} \quad \zeta_0 = 0 .
\]

(Proof: see Homework 5)
Matrix logarithms

For all matrices $B \in \mathbb{R}^{n \times n}$ satisfying $\|B - I\|_2 < 1$ the matrix logarithm is defined as

$$Z(t) = \log(B) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} [B - I]^i.$$ 

Properties:
- The sequence converges uniformly if $\|B - I\|_2 < 1$.
- $\log(e^{tA}) = tA$ for any square matrix $A$ (with $|t|$ sufficiently small).

Interpolation of linear discrete-time systems

Question: Can we recover a linear time-invariant system from its discrete-time variant?

$$\dot{x}(t) = Ax(t) \quad \leftrightarrow \quad \chi_{k+1} = A\chi_k \quad \text{with} \quad A = e^{Ah}$$

Assumption: equidistant step-size $h > 0$.
- In general, there is no unique recovery possible (Homework).
- If $h$ is sufficiently small we can, in principle, recover the continuous time system matrix as
  $$A = \frac{1}{h} \log(A).$$

In practice, the equation $A = \frac{1}{h} \log(A)$ is never used.
- However, for theoretical derivations, the first order approximation $A \approx \frac{1}{h} (I - A)$ is sometimes useful,
  $$\|A - \frac{1}{h} (I - A)\| = O(h).$$