Model Predictive Control

- Model Predictive Control
- Stability Analysis

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Infinite Horizon Control

We would like to solve

\[
J_\infty(x_0) = \min_{\chi, a} \sum_{k=0}^{\infty} l(\chi_k, a_k)
\]

subject to

\[
\begin{align*}
\chi_{k+1} &= A\chi_k + Ba_k, & k \in \mathbb{N} \\
0 &\geq h(\chi_k, a_k), & k \in \mathbb{N} \\
\chi_0 &= x_0
\end{align*}
\]

- \(x_0\) denotes current state measurement
- Problem: infinite horizon!

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Model Predictive Control (MPC)

Idea:

- As soon as current state measurement \(x_0\) is available, solve

\[
V(x_0) = \min_{\chi, a} \sum_{k=0}^{M-1} l(\chi_k, a_k) + J_\infty(\chi_M)
\]

subject to

\[
\begin{align*}
\chi_{k+1} &= A\chi_k + Ba_k, & k \in \{0, \ldots, M\} \\
0 &\geq h(\chi_k, a_k), & k \in \{0, \ldots, M\} \\
\chi_0 &= x_0
\end{align*}
\]

on a sufficiently large horizon \(N\) and terminal cost \(J_\infty \approx J_\infty\).

- Send an optimal \(u_0\) to the real process.
- Shift the time horizon, \(t \leftarrow t + 1\), and repeat.

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Variants of MPC

- "Tracking MPC": assumes \(h(0, 0) \leq 0\) and that \(l\) is positive-definite and satisfies \(l(0, 0) = 0\); often

\[
l(x, u) = x^TQx + u^TRu
\]

- "Economic MPC": only assumes that \(l\) and \(h\) are continuous.
- Other variants include "Robust MPC" and "Output-Feedback MPC", which take additional process noise and/or measurements errors into account. (not part of this lecture)

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MPC Control-Law

- MPC evaluates control law $\mu(x_0) = u_0$ by solving an optimization problem.
- A MPC controller is called stable, if the closed-loop system
  \[ y_{k+1} = Ay_k + B\mu(y_k) \]
  is stable.
- Problem: Stability is difficult to analyze, as we have in general no explicit expression for $\mu$.

Terminal Equilibrium Constraints

For tracking MPC problem, one way to enforce stability is by choosing the end-term
\[ \hat{J}_\infty(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{otherwise} \end{cases} \]
This is equivalent to add the terminal constraint $\chi_N = 0$.

Pros and Cons of Terminal Constraints

- Easy way to enforce and prove stability; “mathematically clean” framework.
- But:
  1. The terminal constraint can lead to infeasibility, although the system is stabilizable.
  2. The terminal constraint can lead to loss of control performance.

Terminal Equilibrium Constraints

Main argument for the introducing terminal constraint $\chi_N = 0$:

- If $u_0, u_1, u_2, \ldots, u_{M-1}$ is an optimal solution of the current problem, then the shifted sequence
  \[ u_1, u_2, \ldots, u_{M-1}, 0 \]
  is a feasible control sequence for the next problem, generating same objective value (we assume $l(0, 0) = 0$ and $h(0, 0) \leq 0$).
- Consequently, we have
  \[ V(y_{k+1}) \leq V(y_k) - l(\chi_0, u_0) \text{.} \]
- If $l$ continuous and positive definite, $V$ is a Lyapunov function.
Infinite Horizon LQR as End-Cost

if \( l(x,u) = x^T Q x + u^T R u, \quad Q \succ 0, \quad R \succ 0, \) is a quadratic tracking objective, we compute solution of ARE

\[
P = A^T P A + Q - A^T P B \left( R + B^T P B \right)^{-1} B^T P A
\]

and use the quadratic terminal cost \( \hat{J}_\infty(x) = x^T P x; \) local stability can be established if the system is controllable, and if no constraints are active in the neighborhood of the steady state.

another option is to use to terminal cost at all—in this case stability can sometimes be established if \( N \) is sufficiently large.