Controllability and Stabilizability of Linear System

Reachable Sets

Reachable points

Let $x_0 \in \mathbb{R}^n$ be given. Consider

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad \text{with} \quad x(0) = x_0.$$ 

A point $x_T \in \mathbb{R}^n$ is called reachable from the point $x_0$ in time $T$, if there exists a control input $u$ with $u(t) \in U$ such that $x(T) = x_T$.

Reachable sets

The set of reachable points at time $t \geq 0$ can be written in the form

$$S(t) = \left\{ G(t,0)x_0 + \int_0^t G(t,\tau)B(\tau)u(\tau)\,d\tau \mid \forall \tau \in [0,t], \, u(\tau) \in U \right\}.$$ 

Here, $G(t,\tau)$ denotes the fundamental solution and $U \subseteq \mathbb{R}^n_u$ the control constraint set.

$S(t)$ can be interpreted as the set of all points $x_T \in \mathbb{R}^n$ to which we can steer the dynamic system.

Example

For the scalar LTI system

$$\dot{x}(t) = ax(t) + bu(t)$$

with control bounds $U = [-1, 1]$, the reachable set $S(t)$ is an interval,

$$S(t) = \left[ e^{at}x_0 + \int_0^t e^{a(t-\tau)}b\,d\tau, \, e^{at}x_0 - \int_0^t e^{a(t-\tau)}b\,d\tau \right].$$
Properties of reachable sets

1. If the set $U$ is bounded, then the set $S(t)$ is for every given $t$
   bounded.
2. If the set $U$ is point symmetric, then the set $S(t)$ is point
   symmetric.
3. If the set $U$ is convex, then the set $S(t)$ is convex.
4. If the set $U$ is convex and compact in $\mathbb{R}^{n_x}$, then the set $S(t)$ is
   convex and compact in $\mathbb{R}^{n_x}$.

Reachable set for unconstrained linear systems

If $U = \mathbb{R}^{n_u}$, the reachable sets of linear systems can be
characterized explicitly (assume $x_0 = 0$).

- If $s \neq 0$ is in $S(t)$, then we have $\alpha s \in S(t)$.
- If $0 \neq s_1, s_2 \in S(t)$ can be reached with controls $u_1, u_2$, then
  $s_1 + s_2 \in S(t)$ can be reached with $u_1 + u_2$.

Putting these two properties together, we know that $S(t)$ must be
a vector space: there must exist a (potentially rank-deficient)
matrix $P(t) \in \mathbb{R}^{n_x \times n_x}$ such that

$$S(t) = \{ P(t)v \mid v \in \mathbb{R}^{n_x} \}.$$  

How can we find/compute such a matrix $P(t)$?

Reachable set for unconstrained linear systems

Controllability and Stabilizability of Linear System

Controllability Grammian

Idea: show that the matrix

$$P(t) = \int_0^t G(t, \tau)B(\tau)B(\tau)^TG(t, \tau)^T d\tau,$$

has the desired properties.

Step 1: The point $P(t)v$ with $v \in \mathbb{R}^{n_x}$ can be reached using the
input $u(\tau) = B(\tau)^TG(t, \tau)^Tv$,

$$x(t) = \int_0^t G(t, \tau)B(\tau)u(\tau) d\tau = P(t)v$$

Thus, $S(t) \supset \{ P(t)v \mid v \in \mathbb{R}^{n_x} \}$. 
Controllability Grammian

**Step 2:** Assume that we can reach a point \( s \notin \{ P(t)v \mid v \in \mathbb{R}^{n_x} \} \).

In this case we can find a vector \( c \) with \( c^T s \neq 0 \) but \( P(t)c = 0 \),

\[
0 < (c^T s)^2 = \left( \int_0^t c^T G(t, \tau) B(\tau) u(\tau) \, d\tau \right)^2 
\leq \left( \int_0^t c^T G(t, \tau) B(\tau) B(\tau)^T G(t, \tau) \, d\tau \right) \int_0^t \| u(\tau) \|^2 \, d\tau 
= c^T P(t) c \int_0^T \| u(\tau) \|^2 \, d\tau 
= 0.
\]

This is a contradiction. Thus, \( S(t) \subseteq \{ P(t)v \mid v \in \mathbb{R}^{n_x} \} \).

Controllability ellipsoids

The set

\[
S(t) = \left\{ \int_0^t G(t, \tau) B(\tau) u(\tau) \, d\tau \mid u : [0, t] \to \mathbb{R}^{n_u}, \int_0^t u(\tau)^2 \, d\tau \leq 1 \right\},
\]

is called the controllability ellipsoid at time \( t \). It can be written as

\[
S(t) = \left\{ P(t)^{\frac{1}{2}} c \mid c \in \mathbb{R}, \, c^T c \leq 1 \right\}.
\]

(This follows from Cauchy-Schwarz inequality for \( L_2 \)-scalar products.)

Lyapunov differential equation

The matrix

\[
P(t) = \int_0^t G(t, \tau) B(\tau) B(\tau)^T G(t, \tau) \, d\tau,
\]

can alternatively be computed from the (inhomogeneous) Lyapunov differential equation

\[
\dot{P}(t) = A(t) P(t) + P(t) A(t)^T + B(t) B(t)^T \quad \text{with} \quad P(0) = 0.
\]

Feedback control

**Idea:** measure \( x(t) \) and “react”, i.e., adjust \( u \) in dependence on state.

- the feedback control law \( \mu : \mathbb{R} \times \mathbb{R}^{n_x} \to U \) can be any measurable function.
- closed loop system satisfies

\[
\dot{x}(t) = A(t) x(t) + B(t) \mu(t, x(t)) \quad \text{with} \quad x(0) = x_0.
\]

- In practice, we have uncertainties (if we hadn’t, we could pre-compute \( u \)). However, let’s first analyze the case that no uncertainties are present.
Stabilizability

- A linear control system is called stabilizable, if there exists a feedback law \( \mu : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{U} \) such that there exists for every \( \epsilon > 0 \) a \( \delta > 0 \) for which the closed-loop trajectory \( x(t) \) satisfies \( \|x(t)\| \leq \epsilon > 0 \) for all initial values \( \|x_0\| \leq \delta \).
- If the closed loop trajectories additionally satisfies
  \[
  \lim_{t \to \infty} x(t) = 0 ,
  \]
  the system is called asymptotically stabilizable.

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Controllability of periodic systems

Let \( A, B \) be periodic and let \( P(t) \) denote the unique solution of
\[
\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T \quad \text{with} \quad P(0) = 0 ,
\]

1. The matrix \( P(kT) \) can for any \( k \in \{1, 2, 3, \ldots, \} \) be written as
   \[
P(kT) = \sum_{i=1}^{k-1} G(T, 0)^i P(T) \left( G(T, 0)^T \right)^i .
   \]
2. Use Cayley-Hamilton theorem to show that there exists for every \( t \geq n_x T \) a constant \( \gamma \in (0, 1] \) such that
   \[
   \gamma P(t) \preceq P(n_x T) \preceq P(t) .
   \]

Cayley-Hamilton theorem

Let \( G \in \mathbb{R}^{n \times n} \) be given. There exist constants \( a_0, \ldots, a_{n-1} \in \mathbb{R} \) such that
\[
G^n = a_0 + a_1 G + a_2 G^2 + \ldots + a_{n-1} G^{n-1} .
\]
It's sufficient to show \( p(G) = 0, \) where \( p(s) = \det(sI - G). \)
- Special case: if \( G = \text{diag}(\lambda_1, \ldots, \lambda_n) \) is diagonal:
  \[
p(G) = \begin{pmatrix}
  \lambda_1 - \lambda_1 \\
  & \ddots \\
  & & \lambda_1 - \lambda_n
\end{pmatrix}
\]
  \[
  \ldots \begin{pmatrix}
  \lambda_n - \lambda_1 \\
  & \ddots \\
  & & \lambda_n - \lambda_n
\end{pmatrix} = 0 .
  \]
- If \( G \) is diagonalizable, diagonalize \( p(G) = T \left( \text{diag}(0^s) \right) T^{-1} = 0. \)
- For general proof: Jordan blocks satisfy \( (\lambda I - J)^n = 0. \)

Controllability of periodic systems

In summary: if the solution \( P(t) \) of
\[
\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T \quad \text{with} \quad P(0) = 0 ,
\]
is not positive definite at time \( t = n_x T, \) then the periodic system
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\]
is not controllable at any time \( t. \) This is because \( P(t) \) cannot be positive definite, since
\[
\gamma P(t) \preceq P(n_x T)
\]
for a \( \gamma > 0 \) and all \( t \geq n_x T. \) Otherwise, if \( P(n_x T) \succ 0 \) the periodic system is controllable, i.e., we can reach any point in state-space after at most \( n_x \) periods.
Controllability matrix of periodic systems

If the functions $A$ and $B$ are periodic, $A(t) = A(t + T)$, $B(t) = B(t + T)$, stabilizability can be analyzed by using linear algebra tools. The matrix

$$P_C = P(n_x T) \quad \text{with} \quad P(t) = \int_0^t G(t, \tau)B(\tau)B(\tau)^T G(t, \tau)^TD\tau$$

is called controllability matrix.

- Recall: the periodic system is controllable if and only if the matrix $P_C$ is positive definite.

Stabilizability of periodic systems

**Theorem**

1. A periodic linear control system is stabilizable if and only if the eigenvalues of the projected monodromy matrix $G_p$ are contained in the closed unit disc and all eigenvalues on the unit circle have algebraic multiplicity 1.
2. A periodic linear control system is asymptotically stabilizable if and only if the eigenvalues of the projected monodromy matrix $G_p$ are contained in the open unit disc.

Projected monodromy matrix

In general, the matrix $P_C$ is not invertible, but its pseudo-inverse, denoted by $P_C^\dagger$, always exists. We denote with

$$G_p = (I - P_C P_C^\dagger)G(T, 0)^{n_x}$$

the projected monodromy matrix.

- We have $P_C G_p = 0$.
- If $P_C$ is positive definite, then $G_p = 0$.

Proof of necessity

Denote with (assume $x_0 = 0$)

$$S_C = \{ P_C v \mid v \in \mathbb{R}^{n_x} \}$$

the set of states that are reachable at time points $t \geq n_x T$.

The discrete-time iterates $\chi_k = x(kn_x T)$ satisfy

$$\chi_{k+1} = G(T, 0)^{n_x} \chi_k + \int_0^{n_x T} G(n_x T, \tau)B(\tau)u(kn_x T + \tau)D\tau , \quad \in S_C$$

which implies

$$\chi_{k+1} - G(T, 0)^{n_x} \chi_k \in S_C .$$
Proof of necessity (continued)

Since $S_C$ is spanned by the columns of $P_C$,

\[ \chi_{k+1} - G_p \chi_k \in S_C. \]

Notice that $G(T, 0)S_C \subseteq S_C$, which implies

\[ G_p S_C = (I - P_C P_C^\dagger) G(T, 0)^n S_C = (I - P_C P_C^\dagger) S_C = \{0\}. \]

Proof of sufficiency

We can always construct the control input such that

\[ \forall k \in \mathbb{N}, \quad \chi_{n \cdot k} = G_p^k \chi_0. \]

This sequence remains (asymptotically) stable under the mentioned conditions.

Proof of necessity (continued)

Now, we can use induction to establish

\[ \chi_k - G_p^k \chi_0 \in S_C \]

for all $k \in \mathbb{N}$. Multiply with $I - P_C P_C^\dagger$:

\[ (I - P_C P_C^\dagger) \chi_k = G_p^k \chi_0. \]

We take the norm on both sides:

\[ \|\chi_k\|_2 \geq \|(I - P_C P_C^\dagger) \chi_k\|_2 = \|G_p^k \chi_0\|_2. \]

This inequality proves the necessity of both conditions.

Controllability assumptions in linear control theory

Many linear control theory articles start like

“We assume that the given linear systems is controllable […]”

with the above discussion in mind, read as

“we restrict ourselves to the behavior of the control system inside the invariant reachable set $S_C$ in which the system is—by definition—controllable”

From a control design perspective: “uncontrollable modes are uninteresting, as there is nothing we can do about them”.
Re-scaling time

Recall $P_C = P(n_x T)$. It is possible to construct time-varying periodic system that are not controllable after one but controllable after $n_x$ periods (do this as an exercise!).

- If this ever happens, re-define $T \leftarrow n_x T$
- Summary: simply assume $P(T)$ is positive definite.